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SHALLOW WATER EQUATIONS ON A ROTATING ATTRACTING SPHERE
2. SIMPLE STATIONARY WAVES AND SOUND CHARACTERISTICS

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This paper studies a model of shallow water on a rotating attracting sphere that describes large-scale motions of the planetary atmospheric gases and World ocean water. The propagation of sound perturbations on an equilibrium state is studied. The system of equations for bicharacteristics is integrated in elliptic functions. A description of simple stationary waves is given. It is proved that there exist two types of solutions (supercritical and subcritical) describing gas motion in a spherical zone, so that one of the boundary parallels is a source and the other is a sink. The obtained solutions are interpreted as large-scale circulating cells in the atmosphere.

Key words: *shallow water, motions on sphere, stationary solutions, propagation of sound perturbations, circulating cells.*

INTRODUCTION

A model of shallow water on a rotating sphere describing large-scale motions in the planetary atmospheres and World ocean is presented in [1]. It is assumed that the thickness of the layer of the incompressible continuous medium (air or water) on the surface of a planet is small compared to the radius of the planet, and the motion in the radial direction can therefore be ignored. As noted in [2], such assumptions are valid if rotation has a significant effect on the motion of the medium. Motions with large time scales are considered. In addition, in the case of large-scale geophysical motions, liquid particle trajectory deviate insignificantly from a sphere in the radial direction.

1. FORMULATION OF THE PROBLEM

The proposed model coincides with the equations of gas dynamics on a rotating sphere for the polytropic gas equation of state with an adiabatic exponent $\gamma = 2$ which describes motions on the surface of the sphere independent of the radius $r = \sqrt{x^2 + y^2 + z^2}$. The system of equations written in a noninertial system of coordinates rotating together with the sphere at constant angular velocity Ω_0 has the form [3]

$$\begin{gathered} Dv = w^2 \cot\theta + r_0 w \cos\theta + (1/4) r_0^2 \sin\theta\cos\theta - f_0 h_\theta, \\ Dw = -vw \cot\theta - r_0 w \cos\theta - f_0 (\sin\theta)^{-1} h_\varphi, \\ Dh + (\sin\theta)^{-1} h (w_\varphi + (v \sin\theta)_\theta) = 0, \end{gathered} \tag{1.1}$$

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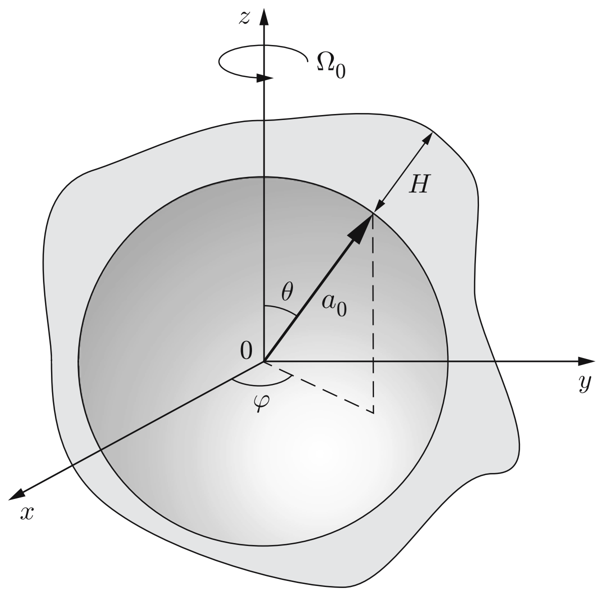

Fig. 1. Formulation of the problem.

where $D = \partial_t + v\,\partial_\theta + (\sin\theta)^{-1} w\,\partial_\varphi$ is the total derivative over the surface of the sphere. Equations (1.1) are written in spherical coordinates: $0 < \theta < \pi$ is the latitude, $0 \leqslant \varphi < 2\pi$ is the longitude; v and w are the meridional and longitudinal velocity components, and $h > 0$ is the depth of the layer. The positive directions are taken to be the directions from north to south and from west to east. The dimensionless parameters r_0 and f_0 are linked to the Rossby number R_0 and Froude number F:

$$R_0 = \frac{V_0}{2a_0\Omega_0}, \qquad F = \frac{V_0}{\sqrt{gH_0}}$$

by the relations

$$r_0 = R_0^{-1}, \qquad f_0 = F^{-2}.$$

Here V_0 and H_0 are the characteristic scales of the velocity component tangential to the sphere and the depth of the layer, a_0 is the radius of the sphere, and g is the acceleration due to gravity (Fig. 1).

The shallow water parameter $\varepsilon = H_0/a_0$ is assumed to be small compared to the parameters r_0 and f_0, which, for the Earth, have the same order of smallness and, hence, rotation and gravitation have comparable effects on the motion of the gas. A description of this motion is the main objective of the present work. A characteristic feature of the model is the compact form of the solution manifold. In [1], infinite-dimensional transformations of the equivalence of system (1.1) in the stationary case are constructed and solutions corresponding to the equilibrium state and zonal flows along the parallels are presented. The problem of the propagation of sound perturbations in the atmosphere for a gas-dynamic model was formulated by Ovsyannikov. In the present paper, we describe sound characteristics on the equilibrium state and study stationary simple waves of system (1.1). We also note papers [3, 4], in which the group properties of some models of atmospheric physics are studied and exact solutions constructed using the symmetry groups admitted by these models are studied.

2. PROPAGATION OF SOUND PERTURBATIONS ON THE EQUILIBRIUM STATE

Since system (1.1) is hyperbolic, the finite velocity of propagation of sound perturbations is of great significance for applications. By virtue of the gas-dynamic analogy, this velocity is equal to the sound velocity $c = \sqrt{f_0 h}$. Let the family of sound characteristics be given by the equations $\chi(t, \theta, \varphi) = \text{const}$. Then, for a given solution $\boldsymbol{u} = (v, w)$, h, the function χ satisfies the equation [5]

$$\chi_t + v\chi_\theta + (\sin\theta)^{-1} w\chi_\varphi = \varepsilon c N \qquad (\varepsilon = \pm 1), \tag{2.1}$$

where

$$N = |\nabla\chi| = \left(\chi_\theta^2 + (\sin\theta)^{-2}\chi_\varphi^2\right)^{1/2}.$$

In this case, the sound perturbations propagate over the surface of a sphere of unit radius.

To derive the equations for the bicharacteristics of systems (1.1) for the given solution, i.e., the characteristics of Eq. (2.1) that are curves in the space $\mathbb{R}^3(\boldsymbol{x})$ along which sound perturbations propagate (sound beams), we use general data on the structure of the characteristics of first-order differential equations [6].

Equation (2.1) is the Hamilton–Jacobi equation for the function $\chi = \chi(t, x_1, \ldots, x_n)$ and is written as

$$\chi_t + H(t, x_1, \ldots, x_n, \chi_1, \ldots, \chi_n) = 0, \tag{2.2}$$

where $\chi_i = \partial\chi/\partial x_i$ $(i = 1, \ldots, n)$. The characteristic system for Eq. (2.2), which is also called the canonical system of differential equations, is written as

$$\frac{dx_i}{dt} = H_{\chi_i}, \qquad \frac{d\chi_i}{dt} = -H_{x_i} \qquad (i = 1, \ldots, n) \tag{2.3}$$

and is a Hamilton system of equations with the Hamiltonian H.

Next, we use the following theorem.

Theorem 1 [6, pp. 115–116]**.** *If, for the differential equation* (2.2)*, the complete integral* $\chi = \varphi(t, x_1, \ldots, x_n, a_1, \ldots, a_n) + a$ *dependent on* $n+1$ *parameters* $a, a_1, \ldots, a_n$ *is known, then the equations*

$$\varphi_{a_i} = b_i, \qquad \varphi_{x_i} = p_i \qquad (i = 1, \ldots, n),$$

where $p_i = \chi_i$*, with* $2n$ *arbitrary parameters* a_i *and* b_i *leads to an implicit* $2n$*-parameter family of solutions of the canonical system of differential equations* (2.3)*.*

Let us study the sound characteristics and bicharacteristics of Eqs. (1.1) for some simple solutions.

The shallow water model (1.1) admits an equilibrium state on which the relative velocity components are equal to zero ($v = w = 0$) and the following depth distribution holds:

$$h = \alpha_0^2(k_0^2 + \sin^2\theta). \tag{2.4}$$

Here $\alpha_0^2 = r_0^2/(8f_0)$, $k_0^2 = 8f_0h_0/r_0^2$, and $h_0 > 0$ are constants. The sound velocity for this solution $c = (r_0/2\sqrt{2})(k_0^2 + \sin^2\theta)^{1/2}$. For $\theta \in (0, \pi)$, the equation

$$r = \alpha_0^2(k_0^2 + \sin^2\theta) \tag{2.5}$$

in space $\mathbb{R}^3(\boldsymbol{x})$ defines a surface of revolution that characterizes a nonspherical equilibrium depth profile. The equilibrium surface (2.5) is shown in Fig. 2 in [1].

Because equation (2.1) is homogeneous in the derivatives of the function χ and the equation of the family of characteristics admits a scale transformation, it follows that, multiplying the function χ by a constant multiplier, we bring the Hamiltonian H for Eqs. (2.1) to the form

$$H = (k_0^2 + \sin^2\theta)^{1/2}\left(\chi_\theta^2 + (\sin\theta)^{-2}\chi_\varphi^2\right)^{1/2}. \tag{2.6}$$

Denoting $Q = (k_0^2 + \sin^2\theta)^{1/2}$, we have $H = QN$.

The Hamilton system (2.3) for the Hamiltonian (2.6) has the form

$$\begin{gathered} \frac{d\theta}{dt} = \frac{Q}{N}\chi_\theta, \qquad \frac{d\varphi}{dt} = \frac{Q}{N}\frac{\chi_\varphi}{\sin^2\theta}, \\ \frac{d\chi_\theta}{dt} = -\frac{\sin\theta\cos\theta}{QN}\left(\chi_\theta^2 - k_0^2(\sin\theta)^{-4}\chi_\varphi^2\right), \qquad \frac{d\chi_\varphi}{dt} = 0. \end{gathered} \tag{2.7}$$

Because $H_t = H_\varphi = 0$, the Hamilton system (2.7) has integrals H and H_φ. The Hamiltonian (2.6) is reduced to the Hamiltonian on the sphere by a change of variables. Indeed, the surface (2.5) of the equilibrium state is conformal to the sphere since its metric has the form

$$dl^2 = Q\,ds^2,$$

where $ds^2 = d\theta^2 + \sin^2\theta\, d\varphi^2$ is the metric on a unit sphere. The conformal correspondence of two-dimensional surfaces in the space $\mathbb{R}^3(\boldsymbol{x})$ can be implemented by two methods: by multiplying the metric by a nonzero multiplier (in this case, by Q) or by a change of coordinates, which is a conformal transformation in the plane of a complex variable that is a stereographic projection of the sphere. In the new variables $\alpha = \alpha(\theta, \varphi)$ and $\beta = \beta(\theta, \varphi)$, we have

$$H_1 = \Big(H_\alpha^2 + (\sin\alpha)^{-2} H_\beta^2\Big)^{1/2}. \tag{2.8}$$

The Hamiltonian (2.8) corresponds to the Hamilton system on the sphere. Hence, the problem of propagation of sound perturbations on the equilibrium state (2.4) is conformally equivalent to the problem of the wave fronts on a sphere [7].

The functions $\alpha(\theta, \varphi)$ and $\beta(\theta, \varphi)$ are complex combinations of elliptic integrals, and transformation to the variables (α, β) does not simplify the problem. In this case, the complexity of the problem lies in searching for and studying the transformation $(\theta, \varphi) \to (\alpha, \beta)$ leading to a Hamiltonian of the form (2.8). The aforesaid suggests that system (2.7) can be integrated in elliptic functions. For this, using Theorem 1, we represent the solution in implicit form. We seek an integral χ of the equation of sound characteristics in the form

$$\chi = -a_0 t - b_0 \varphi + g(\theta),$$

where a_0 and b_0 are parameters; the function g is determined after its substitution into the Hamilton–Jacobi equation (2.2) with the Hamiltonian (2.6.) The calculations yield

$$g(\theta) = \varepsilon\sigma_0 \int\limits_{\theta_0}^{\theta} \sin\theta \sqrt{\frac{1 - l_0^2 \sin^2\theta}{k_0^2 + \sin^2\theta}}\, d\theta \qquad (\varepsilon = \pm 1), \tag{2.9}$$

where

$$\sigma_0^2 = a_0^2 - b_0^2 k_0^2 > 0, \qquad l_0^2 = (b_0/\sigma_0)^2. \tag{2.10}$$

The function $g(\theta)$ of the form (2.9) is represented as a combination of elliptic integrals. This expression is cumbersome and is not given here.

The sound characteristics on the equilibrium state are given by the equations

$$\frac{\partial g}{\partial a_0} = t - t_0, \qquad \frac{\partial g}{\partial b_0} = \varphi - \varphi_0. \tag{2.11}$$

The constants θ_0 and φ_0 determine the point on the unit sphere which is the apex of the characteristic conoid at $t = t_0$.

After substitution of the function g in the form (2.9), (2.10), Eqs. (2.11) become

$$\begin{gathered} \frac{a_0}{|\sigma_0|} \int\limits_{\theta_0}^{\theta} \sin\theta \sqrt{\frac{k_0^2 + \sin^2\theta}{1 - l_0^2 \sin^2\theta}}\, d\theta = t - t_0, \\ \frac{b_0}{|\sigma_0|} \int\limits_{\theta_0}^{\theta} \sin\theta (k_1^2 + \sin^2\theta) \sqrt{\frac{k_0^2 + \sin^2\theta}{1 - l_0^2 \sin^2\theta}}\, d\theta = -(\varphi - \varphi_0), \end{gathered} \tag{2.12}$$

where $k_1^2 = (b_0 k_0)^2$. The integrals on the left side of Eqs. (2.12) are expressed in the form of combinations of elliptic integrals and are not given here because of their bulkiness.

The aforesaid leads to the following theorem.

Theorem 2. *The sound characteristics of the shallow water model* (1.1) *on the equilibrium state* (2.4) *are given implicitly by Eqs.* (2.12) *and are described by combinations of elliptic integrals.*

During integration of the equations of bicharacteristics (2.3), it is necessary to specify the initial data $x_{i0} = x_i(t_0)$ and $\chi_{i0} = \chi_i(t_0)$. Furthermore, the initial values of all derivatives $\chi_{i0} = \chi_i(t_0)$ should be fitted to the initial equation (2.2) and the equations of characteristics $\chi(t_0, x_{10}, \ldots, x_{n0}) = \text{const}$. For the characteristic conoid [the geometrical place of all bicharacteristics from the given point $P_0(\boldsymbol{x}_0)$ at $t = t_0$] the solution of system (2.3) defines a two-parameter family of curves which depends on the parameters λ and μ: $r = R(t, \lambda, \mu)$, $\theta = \Theta(t, \lambda, \mu)$,

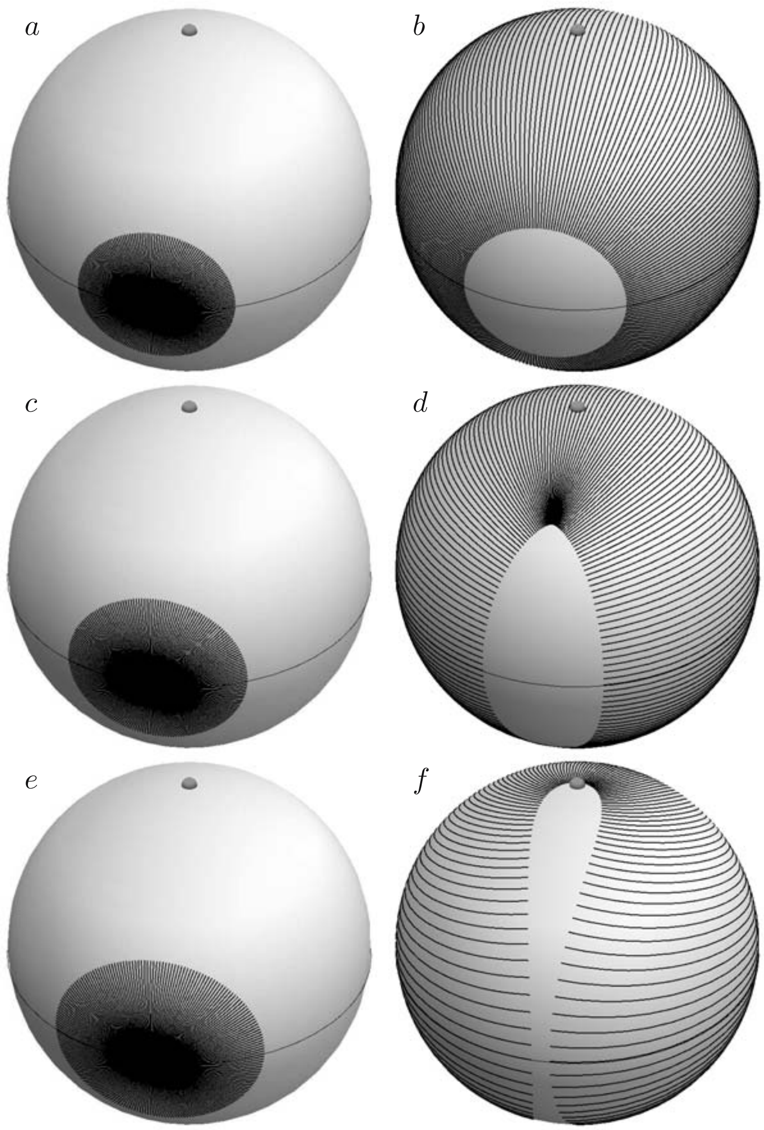

Fig. 2. Characteristic conoid with the apex located at the equator at small times (a, c, and e) and large times (b, d, and f) of perturbation propagation in the absence of rotation (a and b), at moderate rotation velocity ($k_0 > 0.37$) (c and d), and at high rotation velocity ($k_0 < 0.37$) (e and f).

and $\varphi = \Phi(t, \lambda, \mu)$. Examining the perturbation propagation on the surface of the sphere, we obtain a one-parameter family of curves Γ: $\theta = \Phi(t, \lambda)$ and $\varphi = \Phi(t, \lambda)$.

Numerical Analysis of the Characteristic Conoid. From system (2.7), it follows that the form of the characteristic conoid depends on the parameter $k_0^2 = 8f_0h_0/r_0^2$ and the parameters φ_0 and θ_0 which determine the position of the apex of the conoid on the sphere. Since system (2.7) admits translation along φ, the parameter φ_0 is insignificant.

The effect of the parameters k_0 and θ_0 on the formation of caustics is conveniently examined by observing the conoid at large times of perturbation propagation similar to the moment of occurrence of singularities of the perturbation front.

Rotation Effect. In the absence of rotation, if the depth of the liquid layer and the radius of the sphere are fixed, the larger values of k_0 correspond to lower rotation velocities. As $k_0 \to \infty$, the rotation velocity tends to zero.

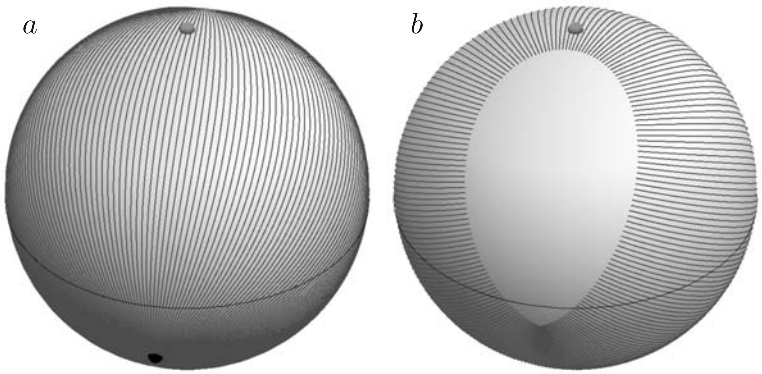

Fig. 3. Conoid with the apex located in the southern hemisphere: (a) top view; (b) opposite side view.

We assume that the apex of the conoid is located at the equator. Then, in the absence of rotation, the liquid depth over the entire sphere is constant and the bicharacteristics are large circles on the sphere. In this case, the perturbation front is always a circle and converges at the point diametrically opposite to the apex of the conoid (Fig. 2a and b). As the rotation velocity increases, the perturbation front is extended along the latitude. It should be noted that, for $k_0 > 0.37$, the caustics at the front due to intersection of the bicharacteristics are formed closer to the poles (Fig. 2c and d), and at smaller k_0 (i.e., at higher rotation velocities), they are formed at the equator (Fig. 2e and f).

For $k_0 = 0$, the equations for the bicharacteristics are integrable in elementary functions:

$$\varphi(t) = t/\sqrt{C_0^2 + 1} + \varphi_0, \quad \theta(t) = \operatorname{arccot}\,(\sinh\tau), \quad \chi_\varphi(t) = \chi_{\varphi 0}, \quad \chi_\theta(t) = -\chi_{\varphi 0} C_0 \cosh\tau.$$

Here $\tau = (C_0/\sqrt{C_0^2 + 1}\,)t + C_1$.

The two limiting cases confirm the behavior of the conoid described above.

Effect of the Position of the Apex of the Conoid. If the apex of the conoid is at the equator, the conoid is symmetric about the equator. If the apex of the conoid is located in the southern hemisphere, the perturbation front converges in the northern hemisphere (Fig. 3). This behavior of the bicharacteristics is natural since the perturbation propagation velocity does not depend on the longitude φ and reaches the maximum value at the equator.

3. SIMPLE STATIONARY WAVES

We consider the simple stationary waves described by system (1.1) in which the required functions v, w, and h depend only on the latitude θ. In this case, Eqs. (1.1) reduces to the system of ordinary differential equations

$$vv' = w^2 \cot\theta + r_0 w \cos\theta + (r_0^2/4)\sin\theta\cos\theta - f_0 h',$$

$$vw' = -vw\cot\theta - r_0 v\cos\theta, \tag{3.1}$$

$$vh'\sin\theta + h(v\sin\theta)' = 0,$$

where the prime denotes the derivative with respect to θ. System (3.1) can be integrated in finite form. This class of solutions was first described in [8].

There are two types of solutions. In the solutions of the first type considered in [1], $v \equiv 0$, and in the solutions of the second type, $v \not\equiv 0$.

We consider the solutions of system (3.1) in which $v \not\equiv 0$. In this case, the second and third equations (3.1) are integrated to give the following representations for the velocity components:

$$v = \frac{q_0}{h\sin\theta}, \qquad w = \frac{w_0}{\sin\theta} - \frac{r_0}{2}\sin\theta \tag{3.2}$$

(q_0 and w_0 are integration constants). The Bernoulli integral for these solutions has the form

$$(v^2 + w^2)/2 + f_0 h - (r_0^2/8)\sin^2\theta = b_0, \tag{3.3}$$

where $b_0 = \text{const}$. Substitution of representations (3.2) into (3.3) yields a algebraic equation of the third degree for the depth:

$$h^3 - \alpha h^2 + \beta = 0. \tag{3.4}$$

Here

$$\alpha = \frac{1}{2f_0}\Big(\varkappa_0 - \frac{w_0^2}{\sin^2\theta}\Big), \qquad \beta = \frac{q_0^2}{2f_0\sin^2\theta}, \qquad \varkappa_0 = 2b_0 + w_0 r_0. \tag{3.5}$$

Solution of Eq. (3.4) yields the depth profile $h = h(\theta)$; substitution of the obtained value of h into representations (3.2) for v yields the velocity vector. Thus, the solution of the problem of simple stationary waves reduces to analysis of Eq. (3.4). According to (3.5), $\beta > 0$. Since, according to the physical meaning, $h > 0$, from (3.4) it follows that $\alpha > 0$. Then, according to (3.5), $\varkappa_0 > 0$.

Let us determine the number of real positive roots of Eq. (3.4). The discriminant of Eq. (3.4) is equal to [9]

$$D_h = \beta(27\beta - 4\alpha^3).$$

Hence, because $\beta > 0$, it follows that for $D_1 = 27\beta - 4\alpha^3 < 0$, Eq. (3.4) has three real roots, and for $D_1 > 0$, it has one real root, and the equation $D_1 = 0$ gives the discriminant curve which defines multiple roots.

According to the Viéte theorem and because $\beta > 0$, the product of the roots of Eq. (3.4) is negative and the sum is positive since $\alpha > 0$. Consequently, one real root is always negative. Hence, the version $D_1 > 0$ can be discarded since it does not lead to physically meaningful solutions.

Equation (3.4) has the sign signature of the coefficients of the form $(+-+)$, and, hence, according to the Descartes theorem [9], it can have two positive real roots, which is possible for $D_1 < 0$. After some calculations, we obtain

$$D_1 = -\frac{\varkappa_0^3}{2f_0^3 s^3}\,P(s), \tag{3.6}$$

where $s = \sin^2\theta$,

$$P(s) = s^3 - 3(\lambda_0 + \mu_0)s^2 + 3\lambda_0^2 s - \lambda_0^3. \tag{3.7}$$

The coefficients of polynomial (3.7) have the form

$$\lambda_0 = w_0^2/\varkappa_0, \qquad \mu_0 = 9f_0^2 q_0^2/\varkappa_0^3. \tag{3.8}$$

From (3.8) it follows that $\lambda_0 > 0$, $\mu_0 > 0$. According to (3.6), $D_1 < 0$ for $P(s) > 0$. The discriminant of the equation $P(s) = 0$ is equal to

$$D_P = 27\mu_0^2\lambda_0^3(4\mu_0 + 9\lambda_0)$$

and is positive for any $\lambda_0 > 0$, $\mu_0 > 0$ (see (3.8)). Hence, the equation $P(s) = 0$ has only one real root. The sign signature of the coefficients of this equation has the form $(+-+-)$. In this signature there are three changes of sign and, hence, according to the Descartes theorem, it has one or three positive roots. Only the first case can take place. In addition, since $s = \sin^2\theta < 1$, $\theta \in (0, \pi)$, this positive root s_* should be in the interval $(0, 1)$, which is provided by the inequalities

$$P(0) = -\lambda_0^3 < 0, \qquad P(1) = (1 - \lambda_0)^3 - 3\mu_0 > 0. \tag{3.9}$$

The first of these inequalities is satisfied automatically, and the second defines the admissible region Γ of parameters in the plane $\mathbb{R}^2(\lambda_0, \mu_0)$.

The interval $I = (\theta_*, \pi - \theta_*)$, where $\sin\theta_* = s_*$, corresponds to the case $D_1 < 0$; hence, in this interval, Eq. (3.4) has two real positive roots $h_1 = h_1(\theta)$ and $h_2 = h_2(\theta)$. The functions h_1 and h_2 can be written by the Cardano formulas [9], but this representation is not effective for the analysis of the solution.

In the space $\mathbb{R}^3(\lambda_0, \mu_0, \theta)$, the region of existence of positive roots of Eq. (3.4) is given by the inequality $P(s) > 0$:

$$\sin^6\theta - 3(\lambda_0 + \mu_0)\sin^4\theta + 3\lambda_0\sin^2\theta - \lambda_0^3 \geqslant 0. \tag{3.10}$$

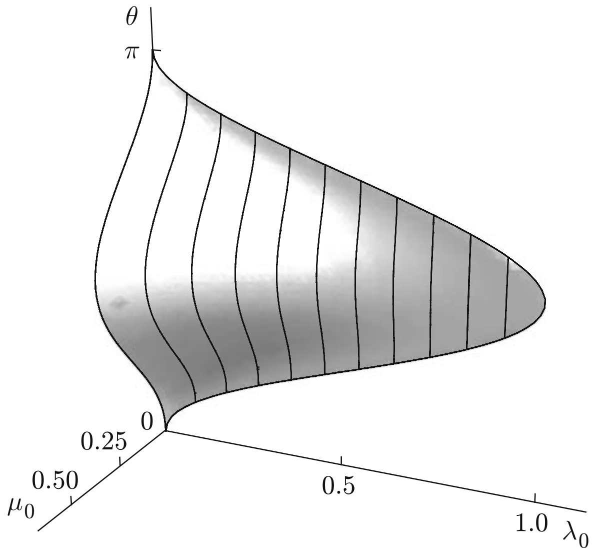

Fig. 4. Region of existence of positive roots of Eq. (3.4).

Inequality (3.10) defines a bounded closed surface in the space $\mathbb{R}^3(\lambda_0, \mu_0, \theta)$ (Fig. 4). For the points inside this surface, the strict inequality (3.10) holds and there are two different positive roots h_1 and h_2. For the points belonging to the surface $P = 0$, a single multiple positive root exists. This surface is the discriminant surface in the space $\mathbb{R}^3(\lambda_0, \mu_0, \theta)$. The concrete parameters $\lambda_0 > 0$ and $\mu_0 > 0$ of the flow correspond to the point $P_0(\lambda_0, \mu_0)$. The segment of the perpendicular from the point P_0 in the region $P > 0$ corresponds to the spherical zone I, and the point of intersections of the perpendicular with the surface $P = 0$ define the boundary parallels Γ_i: $\theta = \theta_i$ $(i = 1, 2)$, where $\theta_1 = \theta_*$ and $\theta_2 = \pi - \theta_*$. These curves are components of the discriminant curve $L = \{\Gamma_i,\ i = 1, 2\}$ on the sphere. All sections of the surface $P = 0$ by the planes $\theta = \text{const}$ are similar.

The aforesaid leads to the following conclusions.

A solution of the form (3.2), (3.4) exists only for some values of the parameters λ_0 and μ_0 given by inequalities (3.9).

The gas flows corresponding to solutions of the form (3.2), (3.4) are defined in a certain spherical zone I symmetric about the equator.

The zone I can be arbitrarily wide except in the regions having the shape of small disks on the poles of the sphere.

For the admissible fixed values of the parameters λ_0 and μ_0, two types of flow exist corresponding to two different positive roots h_1 and h_2 of Eq. (3.4).

For the same width of the zone I, i.e., for a fixed value of θ_*, there are different flows defined by various sets of parameters λ_0 and μ_0.

It remains to clarify the physical meaning of the discriminant curves which are the boundaries of the range of the solution.

4. SOUND CHARACTERISTICS FOR SIMPLE STATIONARY WAVES

We seek the sound characteristics of the stationary equations (1.1) for solutions of the form (3.2) and (3.4). Let these characteristics be given on the sphere by the family of curves $\chi(\theta, \varphi) = \text{const}$. Then, the function χ satisfies the equation

$$v\chi_\theta + (\sin\theta)^{-1} w\chi_\varphi = \varepsilon c N_1, \tag{4.1}$$

where $N_1 = (\chi_\theta^2 + (r\sin\theta)^{-2}\chi_\varphi^2)^{1/2}$; $c = \sqrt{f_0 h}$; v, w, h is a solution of (3.2) and (3.4).

We write the equation of the characteristics resolved for the longitude: $\chi(\theta, \varphi) \equiv \varphi - f(\theta) = \text{const}$. Then, Eq. (4.1) reduces to the equation

$$af'^2 - 2bf' + k = 0, \tag{4.2}$$

where

$$a = (v^2 - f_0 h)\sin^2\theta, \qquad b = vw\sin\theta, \qquad k = w^2 - f_0 h.$$

Equations (4.2) belong to the class of implicit differential equations [10] (usually less accurately referred to as the equations unresolved for the derivative). The main salient feature of these equations is the nonuniqueness of the solution and the fact that the solutions are defined only in a certain region whose boundary is the solution branching manifold. It should be noted that investigation of problems of transonic gas dynamics has initiated the development of complete theory of such equations [11].

Equation (4.2) is studied simply enough. For $a \neq 0$, it is resolved in the form

$$f' = \frac{vw \pm \sqrt{f_0 h(v^2 + w^2 - f_0 h)}}{(v^2 - f_0 h)\sin\theta}. \tag{4.3}$$

The region of the solution of Eq. (4.3) is given by the inequality

$$v^2 + w^2 > f_0 h, \tag{4.4}$$

which describes the region of hyperbolicity of the initial stationary system (1.1) for the given solution. Using the Bernoulli integral (3.3), we obtain

$$v^2 + w^2 = 2b_0 + (r_0^2/4)\sin^2\theta - 2f_0 h. \tag{4.5}$$

Comparing (4.4) and (4.5), we find the hyperbolicity condition of the form

$$h < 2(b_0 + (r_0^2/8)\sin^2\theta)/(3f_0). \tag{4.6}$$

If inequality (4.6) is satisfied, the flow is supercritical; otherwise, it is subcritical. Taking into account the presence of the two flow regimes (3.2) and (3.4) corresponding to the roots h_1 and h_2, and the form of these roots, we can draw the following conclusion. The smaller root $h_1 > 0$ of Eq. (3.4) corresponds to supercritical (supersonic) flow, and the larger root h_2 to subcritical (subsonic) flow.

Equation (4.2) is degenerated for $a = 0$, i.e., for $v^2 = f_0 h$. In this case, it describes sound characteristics of the form $\theta = \theta_0$, which are parallels on the sphere. Thus, there are two types of sound characteristics for solutions of the form (3.2), (3.4). This conclusion follows from the general theory of implicit differential equations, namely, from the Cibrario theorem [10]. The parallels $\theta = \theta_i$ specify the discriminant curves on the sphere which define the boundaries of the region of the solution. Each point P of such a curve is the origin of a pair of sound characteristics (4.3) which form a rostrulum — a singularity of the type of Neil's semicubical parabola — at the point P.

It can be proved that the discriminant curve $L = 0$ specifies sound characteristics on the sphere. The equations of the discriminant curve $L = 0$ are given by

$$h^3 - \alpha h^2 + \beta h = 0, \qquad 3h^2 - 2\alpha h = 0. \tag{4.7}$$

For $h \neq 0$, the second equation in (4.7) implies that $\alpha = 3h/2$. Substituting this expression into the first equation in (4.7), for the boundary parallels Γ_i we obtain $h = (2\beta)^{1/3}$, which coincides with the solution of the equation $v^2 = f_0 h$, which specifies sound characteristic of the form $\theta = \theta_i$.

We calculate the derivative h' at the points of the discriminant curve:

$$h' = \frac{\alpha' h^2 - \beta'}{3h(h - 2\alpha/3)} \to \infty \quad \text{at} \quad h \to \frac{2\alpha}{3}. \tag{4.8}$$

According (4.8), on the boundary parallels $L = \{\Gamma_i,\, i = 1, 2\}$, which are components of the discriminant curve L, solution (3.2), (3.4) undergoes a gradient catastrophe. By virtue of the Bernoulli integral (3.4), the functions v, w, and h are bounded. The boundedness of the derivative (4.8) can be treated as the presence of a source and a sink on the parallels Γ_i: the depth h increases with distance from the curve Γ_1, beginning at a certain value and, in contrast, it decreases to this value in a small vicinity of the parallel Γ_2 (Fig. 5).

Thus, the boundaries Γ_i of the spherical zone I in which solution (3.2) is defined, (3.4) are sound characteristics. One of these parallels, for example, Γ_1 is a source for the given flow, and the second Γ_2 is a sink.

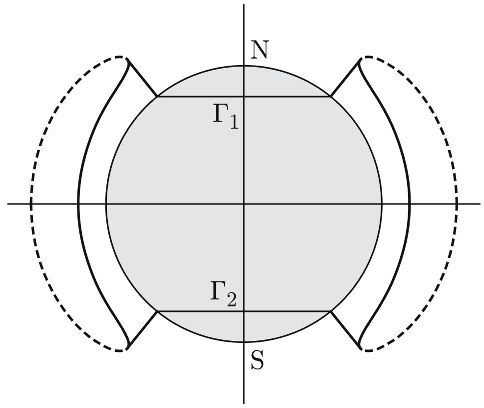

Fig. 5. Typical depth profile on the sphere: solid curves refer to supercritical flow; and dashed curves to subcritical flow.

5. DESCRIPTION OF GAS MOTION

The equation of the flow streamlines

$$\frac{d\theta}{v} = \frac{\sin\theta\, d\varphi}{w}$$

for solution (3.2), (3.4) reduces to the form

$$\varphi = \frac{1}{q_0} \int\limits_{\theta_0}^{\theta} \frac{h(\theta)(w_0 - (r_0/2)\sin^2\theta)}{\sin\theta}\, d\theta, \tag{5.1}$$

where $(0, \theta_0)$ is the starting point on a boundary parallel, for example, Γ_1 that is the origin of the streamline (5.1). By virtue of the rotational symmetry of the solution, any stream line is obtained from the streamline (5.1) by its rotation though the angle φ_0. Figure 6 gives the flow patterns obtained by numerical integration of Eq. (5.1) for various values of the parameter $r_0/(2w_0)$.

The presence of two solutions corresponding to the two roots h_1 and h_2 of Eq. (3.3) allows the solution to be constructed in the form of a cell. Motion, for example, supercritical motion (the smaller root h_1) starts from the parallel of the source Γ_1 and ceases on the parallel Γ_2 corresponding to the sink. In turn, the parallel Γ_2 is the source for the subcritical flow corresponding to the larger root h_2; in this case, the gas moves in the opposite direction flowing into the sink located along the parallel Γ_1.

For $w_0 > 0$, a feature of the solution is that the circumferential velocity components w can vanish on some parallels $\theta = \theta_0$ and $\pi - \theta_0$ symmetric about the equator. According (3.2), this occurs for $2w_0/r_0 < 1$ for values of θ_0 that are solutions of the equation

$$\sin\theta_0 = \sqrt{2w_0/r_0}. \tag{5.2}$$

If Eq. (5.2) has a solution, the circumferential velocity component changes sign in passing through the parallels $\theta = \theta_0$ and $\pi - \theta_0$, and the flow direction along the longitude is reversed (from west to east or vice versa).

6. FORMATION OF TWO SPHERICAL ZONES

In the limiting case where the parameter $\mu = r_0/b_0$ is small, the region of the solution can be divided into two spherical zones.

Omitting the term with μ^2 in the Bernoulli integral (3.3), we obtain

$$v^2 + w^2 + h = 1.$$

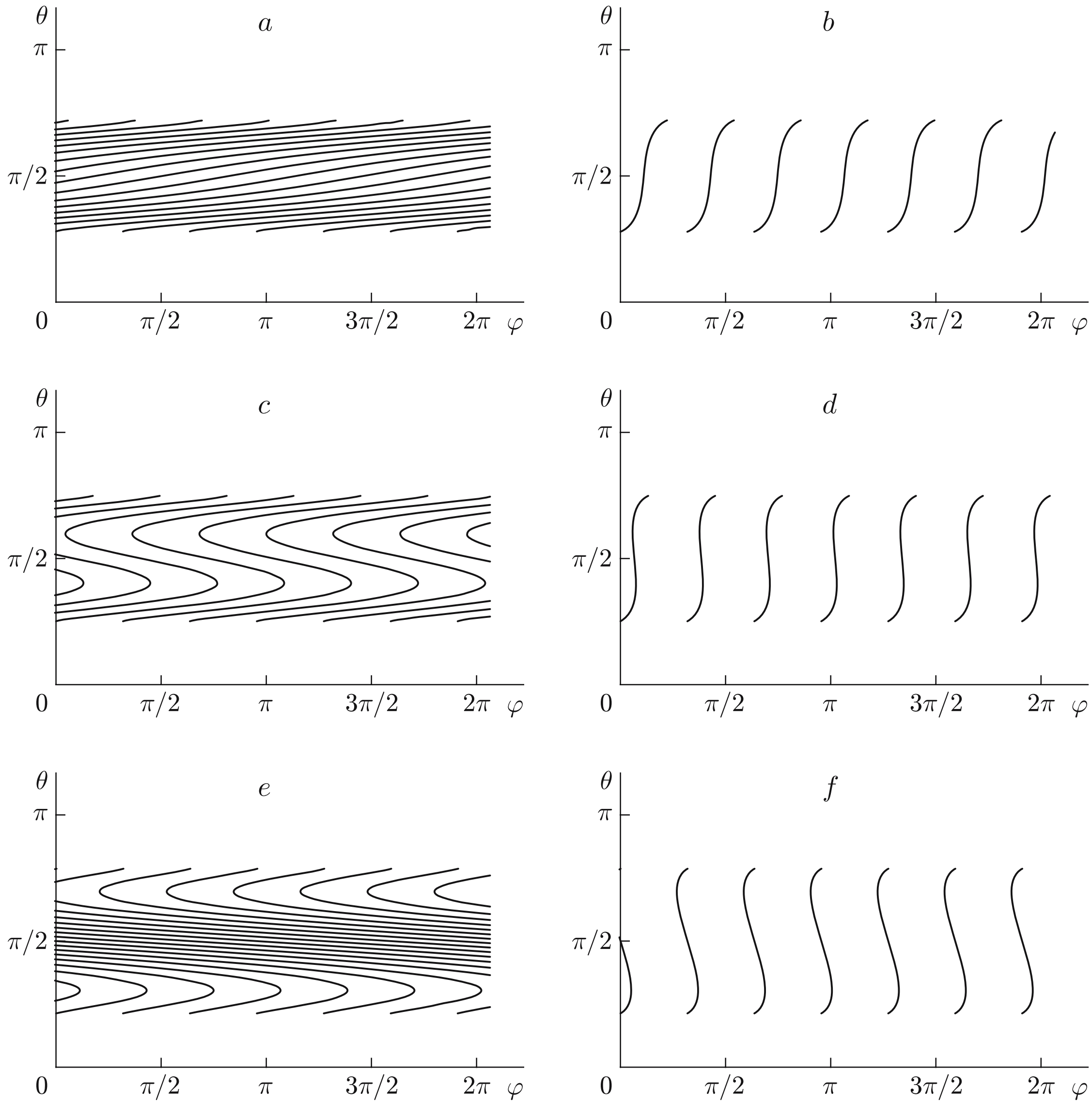

Fig. 6. Streamlines for the subcritical regime (a, c, and e) and supercritical regime (b, d, and f): $r_0/(2w_0) = 0.9$ (a and b), 1.1 (c and d), and 1.5 (e and f).

Retaining the former notation of the functions under the assumption that the corresponding stretching has already been performed, we obtain the following representations for the velocities:

$$v = \frac{v_0}{h \sin\theta}, \qquad w = \frac{w_0}{\sin\theta} + \beta_0 \sin\theta.$$

Equation (3.4) becomes

$$h^3 - \Big(1 - \Big(\frac{w_0}{\sin\theta} + \beta_0 \sin\theta\Big)^2\Big) h^2 + \frac{v_0^2}{\sin^2\theta} = 0. \tag{6.1}$$

In the space $\mathbb{R}^3(v_0, w_0, \theta)$, the inequality

$$-64\sin^6\theta + (432v_0^2 + 192w_0^2)\sin^4\theta - 192w_0^4\sin^2\theta + 64w_0^6 + (192w_0\sin^6\theta - 384w_0^3\sin^4\theta + 192w_0^5\sin^2\theta)\beta_0$$
$$+ (48\sin^8\theta - 288w_0^2\sin^6\theta + 240w_0^4\sin^4\theta)\beta_0^2 + (-96w_0\sin^8\theta + 160w_0^3\sin^6\theta)\beta_0^3$$
$$+ (-12\sin^{10}\theta + 60w_0^2\sin^8\theta)\beta_0^4 + (12w_0\sin^{10}\theta)\beta_0^5 + (\sin^{12}\theta)\beta_0^6 < 0 \tag{6.2}$$

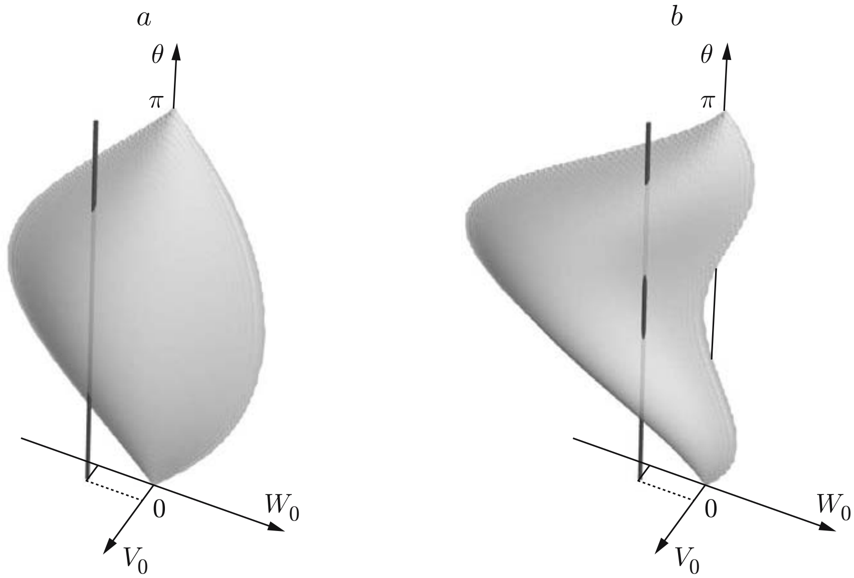

Fig. 7. Surface S for various values of the parameter β_0: (a) $\beta_0 < \beta_0^k$; (b) $\beta_0^k < \beta_0 < \beta_0^c$.

defines the region of existence of two real positive roots of Eq. (6.1) [an analog of the region given by inequality (3.10) in the space $\mathbb{R}^3(\mu_0, \lambda_0, \theta)$]. In this case, the following statement plays an important role.

Lemma 1. *The replacement $w_0 \to w_0 - (\beta_0/2)\sin^2\theta$ reduces inequality* (6.2) *to the inequality*

$$-4\sin^6\theta + (27v_0^2 + 12w_0^2)\sin^4\theta - 12w_0^4\sin^2\theta + 4w_0^6 < 0.$$

The same inequality is obtained from (6.2) *by setting $\beta_0 = 0$.*

The geometrical meaning of Lemma 1 is as follows: in the space $\mathbb{R}^3(v_0, w_0, \theta)$, the region of existence of the solution for a rotating sphere is obtained from the region of existence of the solution for a motionless sphere by a shift of the sections $\theta = \text{const}$ along the axis w_0 under the law specified in the lemma. In this case, the poles $\theta = 0$ and $\theta = \pi$ remain motionless.

Let $v_0 \neq 0$ and $w_0 < 0$. As the parameter β_0 increases, the shape of the surface S given by equality (6.2) changes:

1) for small values of β_0, the spherical zone I is enlarged with increasing β_0;

2) for some value of $\beta_0 = \beta_0^k$, the surface S becomes nonconvex in the θ direction;

3) with a further increase in the parameter β_0, the flow region breaks up into two spherical zones I_1 and I_2 located in different hemispheres symmetrically about the equator;

4) next, as the values of β_0 increase, the spherical zones I_1 and I_2 are shifted toward the poles and their width decreases. For a certain critical value of the parameter $\beta_0 = \beta_0^c$, the zones disappear and the model ceases to work.

Figure 7 shows the surface S for various values of β_0.

A solution in the form of a circulating cell was constructed in Sec. 5. In the presence of two zones in different hemispheres, for which the solution is defined, it is possible to construct flow in the form of symmetric circulating cells (Fig. 8).

Such pairs of cells model the Hadley and Ferrel cells and polar cells [12] which are large-scale (planetary) flows providing atmospheric air circulation, predominantly in the meridional direction. From the region of the equator, where the pressure is lowered, air moves to the region of tropics with a higher pressure (to north and south for the northern and southern cells, respectively). Once a certain maximum pressure is reached, oppositely directed flow arises, in which the pressure decreases in approaching the equator. Rotation leads to deviation of the streamlines from the meridional direction. In the shallow water model, these two flows can be imagined as occurring at different depths. Indeed, from the analysis of the characteristics (see Sec. 4), it follows that supercritical flows are defined for smaller h and subcritical flows for greater h. Thus, for the equatorial cells, there is supercritical

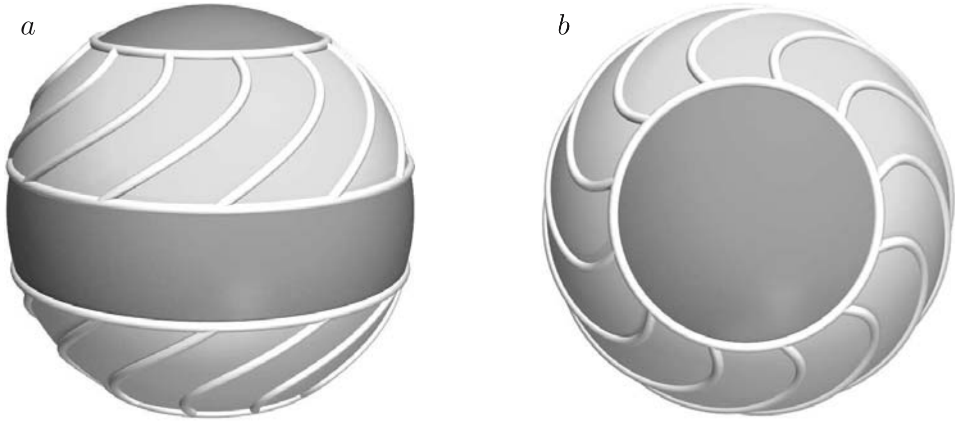

Fig. 8. Symmetric circulating cells: (a) side view; (b) view from the pole.

gas flow to the north in the lower sublayer of the cell and subcritical flow in the opposite direction in the upper sublayer.

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